Hochschule Niederrhein
University of Applied Sciences

Elektrotechnik
und Informatik
Faculty of Electrical Engineering
and Computer Science

An Inequality for Negative Norms with Application to Errors of Finite Element Methods

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Technischer Bericht Nr. 2015-02

IMPRESSUM

Technische Berichte des Fachbereichs Elektrotechnik und Informatik,
Hochschule Niederrhein

ISSN 2199-031X

HERAUSGEBER

Christoph Dalitz und Steffen Goebbels
Fachbereich Elektrotechnik und Informatik

ANSCHRIFT

Hochschule Niederrhein
Reinarzstr. 49
47805 Krefeld

http://www.hsnr.de/fb03/technische-berichte/

S. Goebbels: „An Inequality for Negative Norms with Application to Errors of Finite Element Methods“
Technischer Bericht Nr. 2015-02, Hochschule Niederrhein,
Fachbereich Elektrotechnik und Informatik, 2015

An Inequality for Negative Norms with Application to Errors of Finite Element Methods

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Abstract

This paper deals with an inequality for "negative" norms that can be used to get lower estimates for the error of a finite element discretization of elliptic partial differential equations. It is well known that, if the error is measured in negative norms, additional regularity of a differential equation yields higher convergence orders of finite element methods. Additional regularity means that, depending on given data, solutions are smoother than required by the order of derivatives in a weak formulation of the equation. Obviously, the pointwise error is not affected by the selection of a norm, but negative norms allow for function values to cancel out each other. Thus, negative norms tend to ignore noise. The presented inequality extends sharpness of standard error bounds to such negative norm estimates. It generalizes the "pollution effect" that has been described by L. B. Wahlbin in P. G. Ciarlet and J. L. Lions (Eds): *Handbook of Numerical Analysis, II*, North-Holland, Amsterdam, 1991.

1 Introduction

Typically, the error of a finite element discretization is measured in an L^2 norm, i.e. its square is integrated over the the given domain of the equation, then the square root is taken. For special purposes however, like superconvergence results, supremum-norm bounds or local error estimates, it is important to discuss the error in negative norms (cf. [16, pp. 67] and the literature cited there). Negative norms are norms of dual spaces of Sobolev spaces. A Sobolev space of order s is a function space that contains functions that have (weak) partial derivatives up to an order s. The norm of its dual space, the space of bounded linear functionals that map elements of the Sobolev space to scalars, is called negative norm. Since it is a norm, it has to be non-negative. But this norm turns out to behave like a Sobolev norm with negativ order $-s$. That is the reason for the strange name.

Nitsche's trick (cf. [6, p. 141]) yields a full additional order of convergence when deriving an L^2 error bound from an original finite element error estimate that is given in a Sobolev norm. By replacing the L^2 norm with "negative" norms, it is possible to gain further orders of convergence. The so called "pollution effect" limits convergence rates and shows that the estimate for the one negative norm with the highest convergence rate is best possible. We expand this to the whole scale of negative norms. Although the outcome is not surprising, it extends existing results concerning sharpness of error bounds.

2 Error bounds for finite element discretization

Let $\Omega \subset \mathbb{R}^n$ be an open domain. We additionally assume the domain to be polygonal so that it can be easily triangulated in order to define finite element spaces.

$W^{s,2}(\Omega)$ denotes the Sobolev space (cf. [1]) of real-valued functions f which possess weak derivatives $D^\alpha f$ up to the order s, i.e. $|\alpha| \le s$, belonging to the Hilbert space $L^2(\Omega)$ of square integrable functions on Ω. We use the notations

$$D^\alpha f(x_1,\dots,x_n) = \frac{\partial^{|\alpha|} f(x_1,\dots,x_n)}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}},$$

$$|\alpha| := \sum_{k=1}^{n} \alpha_k,\ \alpha_k \in \mathbb{N}_0 := \{0,1,2,\dots\}.$$

The Sobolev space $W^{s,2}(\Omega)$ is equipped with the inner product

$$(f,g)_{s,2,\Omega} = \sum_{|\alpha|\leq s} (D^\alpha f, D^\alpha g)_{L^2(\Omega)},$$
$$(f,g)_{L^2(\Omega)} = \int_\Omega f\cdot g\, d(x_1,\ldots,x_n).$$

Thus, $W^{s,2}(\Omega)$ is a Hilbert space with norm $\|f\|_{s,2,\Omega} = \sqrt{(f,f)_{s,2,\Omega}}$. To take care of boundary conditions, we denote $W_0^{1,2}(\Omega)$ as the closure of $C_{00}^\infty(\Omega)$ in $W^{1,2}(\Omega)$, where $C_{00}^\infty(\Omega)$ is the set of infinitely often differentiable functions with compact support in Ω. That means that functions in $W_0^{1,2}(\Omega)$ can be regarded as zero on the boundary of Ω.

Finite element methods are discretizations of differential equations that are posed as weak problems. One derives a weak problem from a differential equation, given as a boundary value problem, by multiplying the equation with test functions and then by integrating the products using partial integration. We consider the weak problem to find a solution $u \in W_0^{1,2}(\Omega)$ satisfying

$$a(u,w) = f^*(w) \text{ for all } w \in W_0^{1,2}(\Omega), \qquad (1)$$

where $a(\cdot,\cdot)$ is a bounded, elliptic bilinear form, i.e.

$$|a(u,w)| \leq C\|u\|_{1,2,\Omega}\|w\|_{1,2,\Omega},\, a(u,u) \geq c\|u\|_{1,2,\Omega}^2$$

for all $u,w \in W_0^{1,2}(\Omega)$, and f^* is a bounded linear functional on $W^{1,2}(\Omega)$, i.e. $f^* \in W^{1,2}(\Omega)^*$. For example, a typical choice for $a(u,w)$ would be

$$\sum_{|\alpha|,|\beta|\leq 1} \int_\Omega a_{\alpha,\beta}(x_1,\ldots,x_n)(D^\alpha u(x_1,\ldots,x_n))\cdot$$
$$\cdot (D^\beta w(x_1,\ldots,x_n))\, d(x_1,\ldots,x_n). \qquad (2)$$

We assume that the coefficient functions $a_{\alpha,\beta}$ are essentially bounded and Lipschitz continuous for $|\alpha| = |\beta| = 1$. Also, they have to be chosen such that $a(\cdot,\cdot)$ becomes elliptic.

The theorem of Lax and Milgram (cf. [4, p. 62]) states that there is a unique solution u for each f^*. If the functional is given by $f^*(w) := (f,w)_{L^2(\Omega)}$ with $f \in W^{\nu,2}(\Omega)$, $\nu \in \mathbb{N}_0$, then under reasonable conditions on $a(\cdot,\cdot)$ and on the boundary of Ω, elliptic regularity additionally implies $u \in W^{\nu+2,2}(\Omega)$ (cf. [11, p. 200], [8, p. 187] or [11, p. 197], see for example [7, p. 160] for convex polygonal domains in connection with the Laplacian).

A finite element discretization computes unique approximate solutions u_h that belong to finite element subspaces $V_h \subset W_0^{1,2}(\Omega)$ and fulfill

$$a(u_h,w_h) = f^*(w_h) \text{ for all } w_h \in V_h. \qquad (3)$$

Typically, the functions in V_h are continuous piecewise polynomials defined on a triangulation of a polygonally bounded domain Ω into finite elements. Then often parameter $h \geq 0$ denotes both a lower bound $c\cdot h$ and an upper bound $C\cdot h$ for the diameters of finite elements, where C is the diameter of Ω. Here we assume such bounds for the spaces V_h.

Céa's lemma allows to estimate the finite element error by an error of best approximation (cf. [6, p. 113]):

$$\|u-u_h\|_{1,2,\Omega} \leq C \inf_{w\in V_h} \|u-w\|_{1,2,\Omega}.$$

If V_h consists of piecewise polynomials of order r, one can expect the error to decrease at most like h^r depending on the smoothness of the solution u (Taylor expansion). Nitsche's trick (cf. [6, p. 141], [2, p. 85]) allows to gain an additional order of convergence (factor h^1) for a convex domain Ω, if one measures the error in the L^2 norm:

$$\|u-u_h\|_{L^2(\Omega)} \leq Ch\|u-u_h\|_{1,2,\Omega}. \qquad (4)$$

This is not surprising, because the diameter of the finite elements, which is proportional to h, leads to a factor $1/h$ when dealing with first derivatives in V_h. If one changes from $W^{1,2}$ to L^2 norm, first derivatives are no longer considered.

By replacing the L^2-norm with a "negative norm", one obtains further additional orders of convergence.

Sobolev spaces of negative order are dual spaces (see [4, p. 40], cf. [1, pp. 62–65]): For $\nu \in \mathbb{N}_0 := \{0,1,2,\ldots\}$ one defines $(1/p+1/q=1)$

$$W^{-\nu,p}(\Omega) := (W^{\nu,q}(\Omega))^*.$$

We are only interested in the case $p=q=2$, where $L^2(\Omega)$ is continuously embedded in $W^{-\nu,2}(\Omega)$ via $f \in L^2(\Omega) \mapsto (f,\cdot)_{L^2(\Omega)} \in W^{-\nu,2}(\Omega)$, because

$$\|f\|_{-\nu,2,\Omega} := \|(f,\cdot)_{L^2(\Omega)}\|_{-\nu,2,\Omega}$$
$$:= \sup_{0\neq g\in W^{\nu,2}(\Omega)} \frac{(f,g)_{L^2(\Omega)}}{\|g\|_{\nu,2,\Omega}}$$

$$\leq \sup_{0\neq g\in W^{\nu,2}(\Omega)} \frac{\|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)}}{\|g\|_{\nu,2,\Omega}} \leq \|f\|_{L^2(\Omega)}.$$

Therefore, we can measure L^2 functions in negative norms $\|\cdot\|_{-\nu,2,\Omega}$. For $\nu = 0$ we can choose $g = f$ to see that $\|f\|_{0,2,\Omega} = \|f\|_{L^2(\Omega)}$.

Negative norms naturally occur if one discusses powers T^ν of the solution operator T of the corresponding elliptic boundary value problem. T maps an inhomogeneity $f \in L^2(\Omega)$ to its solution u. Under certain preliminaries, an inner product $(g,w)_{-\nu} := (T^\nu g, w)_{L^2(\Omega)}$ leads to a norm equivalent to $\|\cdot\|_{-\nu,2,\Omega}$ (cf. [15]).

Finite element methods can be formulated as variational problems. Such problems also occur in methods for the reconstruction of noised images. A reason is that they can help to separate oscillating patterns from pure noise. That makes them interesting when dealing with noised images cf. [12] and the literature cited there.

It is shown in [4, p. 147], [14, pp. 166] or [16, p. 68] for $0 \leq \nu \leq r-1$, under the assumption of certain additional elliptic regularity and approximation properties of the finite element spaces of piecewise polynomials of order r, that (cf. (4))

$$\|u-u_h\|_{-\nu,2,\Omega} \leq Ch^{\nu+1}\|u-u_h\|_{1,2,\Omega}. \tag{5}$$

This estimate is not really surprising. For example,

$$\sup_{0\neq w\in L^2(\Omega)} \frac{(u-u_h,w)_{L^2(\Omega)}}{\|w\|_{L^2(\Omega)}} = \|u-u_h\|_{L^2(\Omega)},$$

where the supremum is reached for $w = u-u_h \in L^2(\Omega)$. If we calculate the supremum over the smaller set of functions $w \in W^{1,2}(\Omega)$ then the value becomes smaller. It becomes even smaller by dividing through the larger norm $\|w\|_{1,2,\Omega}$. Thus, $\|u-u_h\|_{-1,2,\Omega}$ might be significant smaller than $\|u-u_h\|_{L^2(\Omega)}$.

A proof of (5) (cf. [4, p. 147]) uses the unique solution u_w of the adjoint problem

$$a(g,u_w) = (g,w)_{L^2(\Omega)} \text{ for all } g \in W_0^{1,2}(\Omega), \tag{6}$$

where $w \in W^{\nu,2}(\Omega)$ is an arbitrary function that serves as inhomogeneity. Let $(u_w)_h \in V_h$ be the finite element solution of the corresponding discrete problem

$$a(g_h,(u_w)_h) = (g_h,w) \text{ for all } g_h \in V_h.$$

We also note that if u is the weak solution of (1) and u_h its discrete counterpart of (3), then $a(u,g_h) = a(u_h,g_h)$ for each $g_h \in V_h$ such that

$$a(u-u_h,g_h) = 0 \text{ for all } g_h \in V_h. \tag{7}$$

By putting this together and remembering that $a(\cdot,\cdot)$ is bounded, we get

$$\begin{aligned}
\|u-u_h\|_{-\nu,2,\Omega} &= \sup_{0\neq w\in W^{\nu,2}(\Omega)} \frac{(u-u_h,w)_{L^2(\Omega)}}{\|w\|_{\nu,2,\Omega}} \\
&\overset{(6)}{=} \sup_{0\neq w\in W^{\nu,2}(\Omega)} \frac{a(u-u_h,u_w)}{\|w\|_{\nu,2,\Omega}} \\
&\overset{(7)}{=} \sup_{0\neq w\in W^{\nu,2}(\Omega)} \frac{a(u-u_h,u_w-(u_w)_h)}{\|w\|_{\nu,2,\Omega}} \\
&\leq C\|u-u_h\|_{1,2,\Omega} \sup_{0\neq w\in W^{\nu,2}(\Omega)} \frac{\|u_w-(u_w)_h\|_{1,2,\Omega}}{\|w\|_{\nu,2,\Omega}}.
\end{aligned}$$

If the regularity of the problem implies that for $w \in W^{\nu,2}(\Omega)$ one has $u_w \in W^{\nu+2,2}(\Omega)$ with $\|u_w\|_{\nu+2,2} \leq C\|w\|_{\nu,2}$, then a typically fulfilled Jackson-type inequality $\|u_w-(u_w)_h\|_{1,2,\Omega} \leq Ch^{\nu+1}\|u_w\|_{\nu+2,2,\Omega}$ implies $\|u_w-(u_w)_h\|_{1,2,\Omega} \leq Ch^{\nu+1}\|w\|_{\nu,2,\Omega}$, and (5) holds true.

3 Sharpness of negative norm estimates

The question arises whether estimate (5) is best possible with regard to the exponent of h. A lower estimate can be established for symmetric bilinear forms via the so called pollution effect (cf. [17, p. 425], see [4, Section 5.8]). By taking (7) into account, there is

$$\begin{aligned}
(f,u-u_h)_{L^2(\Omega)} &= a(u,u-u_h) = a(u-u_h,u) \\
&= a(u-u_h,u-u_h) \geq c\|u-u_h\|^2_{1,2,\Omega}.
\end{aligned}$$

If we additionally know about the inhomogeneity f that $f \neq 0$ and $f \in W^{\nu,2}(\Omega)$, then

$$\begin{aligned}
\|u-u_h\|_{-\nu,2,\Omega} &= \sup_{0\neq w\in W^{\nu,2}(\Omega)} \frac{(u-u_h,w)_{L^2(\Omega)}}{\|w\|_{\nu,2,\Omega}} \\
&\geq \frac{(u-u_h,f)_{L^2(\Omega)}}{\|f\|_{\nu,2,\Omega}} \\
&\geq c\frac{\|u-u_h\|^2_{1,2,\Omega}}{\|f\|_{\nu,2,\Omega}}.
\end{aligned}$$

This implies $\|u-u_h\|_{-\nu,2,\Omega} \neq o(\|u-u_h\|_{1,2,\Omega}^2)$. If $\|u-u_h\|_{1,2,\Omega}$ behaves like h^r, then $\|u-u_h\|_{-\nu,2,\Omega} \neq o(h^{2r})$ showing the sharpness of (5) for $\nu = r-1$.

The aim of this paper is to also discuss the values $\nu \in \{1,2,\ldots,r-2\}$. This will be done in equation (9) with the following lemma.

Lemma 1 *Let $\Omega \subset \mathbb{R}^n$ be an open, bounded domain with a Lipschitz boundary (that is given for a polygonal domain), $0 \neq f \in W^{1,2}(\Omega)$ and $\nu \in \mathbb{N}$. Then there exists a constant $c>0$, independent of f, such that*

$$\|f\|_{-\nu,2,\Omega} \geq c\|f\|_{L^2(\Omega)}^{1+\nu}/\|f\|_{1,2,\Omega}^{\nu}. \tag{8}$$

Before we prove the lemma, we apply it to the finite element error. In [13, pp. 174], [9, p. 91] (cf. [10]) counter examples $u_\omega \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ are constructed with the help of a quantitative resonance principle. The examples show – not surprisingly – that well-known error estimates for finite element methods are indeed best possible but also can be extended to optimal error bounds in terms of moduli of continuity. The lengthy construction requires an open, polygonal domain Ω, a bilinear form (2) with functions $a_{\alpha,\beta}$ that, for $\beta = 1$, are continuously differentiable on the closure of Ω, and an inhomogeneity $f_\omega^* = (f_\omega,\cdot)_{L^2(\Omega)}$. Finite element spaces V_h are constructed from n-simplexes. Then the finite element error $u_\omega - (u_\omega)_h$ fulfills

$$\|u_\omega - (u_\omega)_h\|_{1,2,\Omega} \leq Ch\omega(h^{r-1}),$$
$$\|u_\omega - (u_\omega)_h\|_{L^2(\Omega)} \geq ch^2\omega(h^{r-1}),$$

where ω is an abstract modulus of continuity, i.e. a function continuous on $[0,\infty)$ such that $0=\omega(0) < \omega(\delta_1) \leq \omega(\delta_1+\delta_2) \leq \omega(\delta_1)+\omega(\delta_2)$, for $0<\delta_1,\delta_2$. The smoothness of the counter examples can be measured with moduli of continuity that behave like these abstract moduli of continuity. The counter examples belong to certain Lipschitz classes if $\omega(\delta) = \delta^\gamma$ for some $0 < \gamma \leq 1$.

From this result we immediately conclude sharpness for $0 \leq \nu \leq r-1$ with (5) and Lemma 1

$$\begin{aligned}\|u_\omega - (u_\omega)_h\|_{-\nu,2,\Omega} &\leq C_1 h^{\nu+1}\|u_\omega - (u_\omega)_h\|_{1,2,\Omega}\\ &\leq C_2 h^{\nu+2}\omega(h^{r-1}),\\ \|u_\omega - (u_\omega)_h\|_{-\nu,2,\Omega} &\geq c\frac{[h^2\omega(h^{r-1})]^{1+\nu}}{(h\omega(h^{r-1}))^\nu}\\ &= ch^{\nu+2}\omega(h^{r-1}).\end{aligned} \tag{9}$$

The rest of the paper is concerned with the proof of Lemma 1. We need to interpret Sobolev spaces as interpolation spaces using the real interpolation method. To this end, let X and U be Banach spaces, equipped with norms $\|\cdot\|_X$ and $\|\cdot\|_U$ such that U is continuously embedded in X. Then the K-functional is defined for $\delta > 0$ and $f \in X$ as

$$\begin{aligned}K(\delta,f,X,U) &= K(\delta,f,(X,\|\cdot\|_X),(U,\|\cdot\|_U))\\ &:= \inf_{w\in U}[\|f-w\|_X + \delta\|w\|_U].\end{aligned}$$

For $0<\Theta<1$ let (cf. [5, p. 168])

$$\|f\|_{[X,U]_{\Theta,2}} = \left(\int_0^\infty t^{-2\Theta}K(t,f,X,U)^2\frac{dt}{t}\right)^{1/2}. \tag{10}$$

Then $[X,U]_{\Theta,2} := \{f \in X : \|f\|_{[X,U]_{\Theta,2}} < \infty\}$ is a Banach space with norm $\|\cdot\|_{[X,U]_{\Theta,2}}$. Especially for $0 \neq f \in U$ there is $K(\delta,f,X,U) \leq \|f\|_X$ and $K(\delta,f,X,U) \leq \delta\|f\|_U$. This yields the well known norm estimate

$$\|f\|_{[X,U]_{\Theta,2}} \leq \left(\int_0^{\|f\|_X/\|f\|_U} t^{-2\Theta}t^2\|f\|_U^2\frac{dt}{t} + \int_{\|f\|_X/\|f\|_U}^\infty t^{-2\Theta}\|f\|_X^2\frac{dt}{t}\right)^{\frac{1}{2}} = \frac{\|f\|_X^{1-\Theta}\|f\|_U^\Theta}{\sqrt{2\Theta(1-\Theta)}}$$

or

$$\|f\|_X \geq \left[\sqrt{2\Theta(1-\Theta)}\frac{\|f\|_{[X,U]_{\Theta,2}}}{\|f\|_U^\Theta}\right]^{\frac{1}{1-\Theta}}. \tag{11}$$

Let $X = (W^{\nu+1,2}(\Omega))^*$ be the dual space of $W^{\nu+1,2}(\Omega)$ equipped with norm $\|\cdot\|_{-\nu-1,2,\Omega}$. As a subspace of X we choose $U = (W^{\nu-1,2}(\Omega))^*$ with norm $\|\cdot\|_{-\nu+1,2,\Omega}$. Then for $f \in L^2(\Omega)$ and $\Theta = 1/2$

$$\|f\|_{-\nu-1,2,\Omega} \geq c\left[\frac{\|f\|_{[(W^{\nu+1,2}\Omega)^*,(W^{\nu-1,2}\Omega)^*]_{\frac{1}{2},2}}}{\|f\|_{-\nu+1,2,\Omega}^{\frac{1}{2}}}\right]^2. \tag{12}$$

The main area of applications for interpolation spaces are Sobolev spaces. Indeed, for $s_1,s_2 \in \mathbb{N}_0$, $s_1 < s_2$, one can show for domains with Lipschitz boundary (cf. [4, Section 14.2])

$$[W^{s_1,2}(\Omega),W^{s_2,2}(\Omega)]_{\Theta,2} = W^{(1-\Theta)s_1+\Theta s_2,2}(\Omega), \tag{13}$$

where norm of interpolation space and Sobolev norm are equivalent. To estimate the norm of dual space interpolation in (12), we cite the duality theorem from

[3, p. 54]: Let $U \subset X$ be continuously embedded and dense in X. Then for $0 < \Theta < 1$ there is

$$([X,U]_{\Theta,2})^* = [U^*,X^*]_{1-\Theta,2}$$

with equivalent norms. Please note that in [3] a slightly different definition of the K-functional is used, where U does not have to be a subset of X and where $K(t,f,X,U) = tK(\frac{1}{t},f,U,X)$. Thus, substitution of $1/t$ in (10) yields the presented form of the duality theorem.

With $X = W^{\nu-1,2}(\Omega)$, $U = W^{\nu+1,2}(\Omega)$, and $\Theta = \frac{1}{2}$ we get

$$\begin{aligned} &\|f\|_{[(W^{\nu+1,2}(\Omega))^*,(W^{\nu-1,2}(\Omega))^*]_{\frac{1}{2},2}} \\ &\geq c\|f\|_{([W^{\nu-1,2}(\Omega),W^{\nu+1,2}(\Omega)]_{\frac{1}{2},2})^*}. \end{aligned}$$

Now we can use the representation (13) of Sobolev spaces as interpolation spaces and get for $s_1 = \nu - 1$ and $s_2 = \nu + 1$:

$$\begin{aligned} \|f\|_{[(W^{\nu+1,2}(\Omega))^*,(W^{\nu-1,2}(\Omega))^*]_{\frac{1}{2},2}} &\geq c\|f\|_{(W^{\nu,2}(\Omega))^*} \\ &= c\|f\|_{-\nu,2,\Omega}. \end{aligned}$$

Therefore, (12) becomes

$$\|f\|_{-\nu-1,2,\Omega} \geq c\|f\|^2_{-\nu,2,\Omega}/\|f\|_{-\nu+1,2,\Omega}.$$

We use the inequality recursively. To this end, let $a_\nu := \|f\|_{-\nu,2,\Omega}$:

$$a_{\nu+1} \geq c\frac{a_\nu^2}{a_{\nu-1}} \geq c^3\frac{a_{\nu-1}^3}{a_{\nu-2}^2} \geq c^6\frac{a_{\nu-2}^4}{a_{\nu-3}^3} \geq \dots,$$

Thus, we find

$$\|f\|_{-\nu-1,2,\Omega} \geq c\|f\|^{1+\nu}_{-1,2,\Omega}/\|f\|^{\nu}_{0,2,\Omega}. \tag{14}$$

In Lemma 1 we assume that $0 \neq f \in W^{1,2}(\Omega)$. With this additional prerequisite, we notice for $\nu = 1$

$$\|f\|_{-1,2,\Omega} \geq \frac{(f,f)_{L^2(\Omega)}}{\|f\|_{1,2,\Omega}} = \frac{\|f\|^2_{L^2(\Omega)}}{\|f\|_{1,2,\Omega}}.$$

This is (8) for $\nu = 1$ and, when used in (14), it also shows (8) for $\nu > 1$.

The result can also be obtained by interpolation between Sobolev spaces and their dual spaces. If one chooses $X = W^{-\nu,2}(\Omega) = (W^{\nu,2}(\Omega))^*$, and $U = W^{1,2}(\Omega)$, then estimate (8) follows by (11) and [4, Theorem 14.2.7, p. 342] that implies $[X,U]_{\Theta,2} = L^2(\Omega)$ for $\Theta = \nu/(1+\nu)$:

$$\|f\|_{-\nu,2,\Omega} \geq c\left(\frac{\|f\|_{L^2(\Omega)}}{\|f\|^{\Theta}_{1,2,\Omega}}\right)^{\frac{1}{1-\Theta}} = c\left(\frac{\|f\|_{L^2(\Omega)}}{\|f\|^{\nu/(1+\nu)}_{1,2,\Omega}}\right)^{1+\nu}.$$

4 Conclusion

Additional convergence rates occur, if finite element errors are measured in negative norms. These additional rates originate from the regularity of the differential equation. They are more or less independent of the smoothness of a given solution. On the other hand, there are known counter examples that show that bounds for the L^2 or $W^{1,2}$ error are best possible. These estimates depend on the smoothness of a solution. By combining both aspects, we have shown that the counter examples also directly prove sharpness of negative norm error bounds, i.e., additional speedup of convergence rates can not be achieved.

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